

On the Complexity of SNP Block Partitioning Under the Perfect Phylogeny Model

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Outline

- 1 Introduction**
 - The Model and the Problem
 - The Integrated Approach
- 2 Bad News: Hardness Results**
 - Hardness of PP-Partitioning of Haplotype Matrices
 - Hardness of PP-Partitioning of Genotype Matrices
- 3 Good News: Tractability Results**
 - Perfect Path Phylogenies
 - Tractability of PPP-Partitioning of Genotype Matrices

What is haplotyping and why is it important?

You hopefully know this after the previous three talks. . .

General formalization of haplotyping.

Inputs

- A **genotype matrix** G .
- The **rows** of the matrix are **taxa / individuals**.
- The **columns** of the matrix are **SNP sites / characters**.

Outputs

- A **haplotype matrix** H .
- Pairs of rows in H **explain** the rows of G .
- The haplotypes in H are **biologically plausible**.

Our formalization of haplotyping.

Inputs

- A genotype matrix G .
- The rows of the matrix are individuals / taxa.
- The columns of the matrix are SNP sites / characters.
- The problem is directed: one haplotype is known.
- The input is biallelic: there are only two homozygous states (0 and 1) and one heterozygous state (2).

Outputs

- A haplotype matrix H .
- Pairs of rows in H explain the rows of G .
- The haplotypes in H form a perfect phylogeny.

We can do perfect phylogeny haplotyping efficiently, but ...

1 Data may be missing.

- This makes the problem NP-complete ...
- ... even for very restricted cases.

Solutions:

- Additional assumption like the rich data hypothesis.

2 No perfect phylogeny is possible.

- This can be caused by chromosomal crossing-over effects.
- This can be caused by incorrect data.
- This can be caused by multiple mutations at the same sites.

Solutions:

- Look for phylogenetic networks.
- Correct data.
- Find blocks where a perfect phylogeny is possible.

How blocks help in perfect phylogeny haplotyping.

- 1 Partition the site set into overlapping contiguous blocks.
- 2 Compute a perfect phylogeny for each block and combine them.
- 3 Use dynamic programming for finding the partition.

Genotype matrix

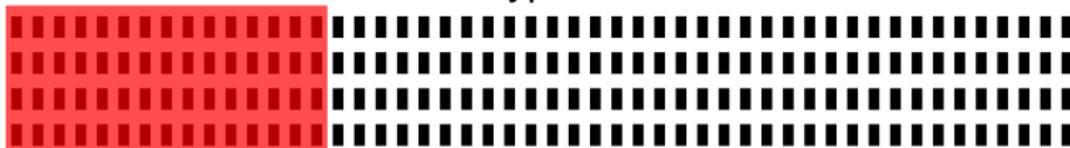


no perfect phylogeny

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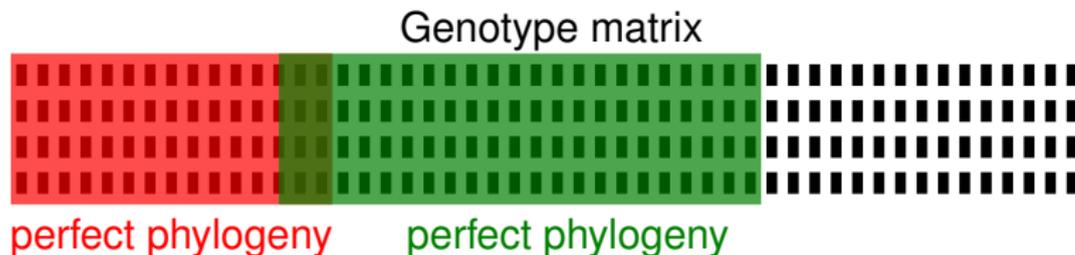
Genotype matrix



perfect phylogeny

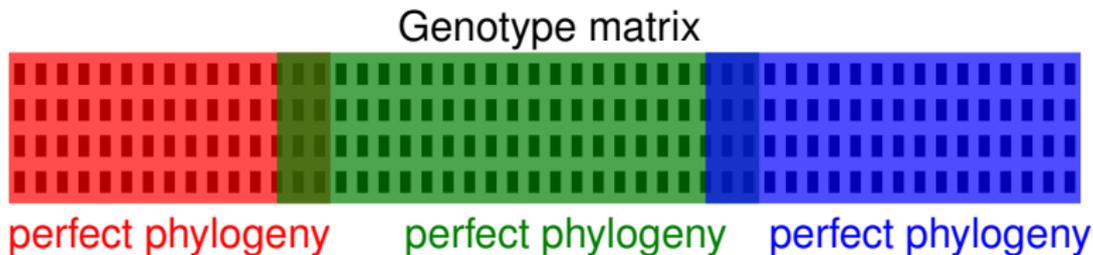
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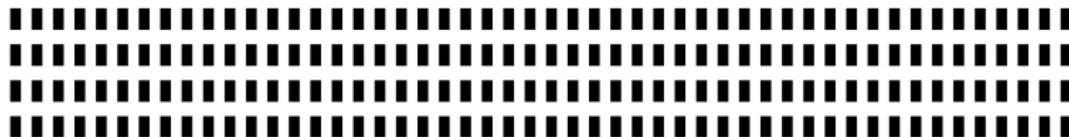
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Objective of the integrated approach.

- 1 Partition the site set into **noncontiguous** blocks.
- 2 Compute a perfect phylogeny for each block and combine them.
- 3 **Compute partition while computing perfect phylogenies.**

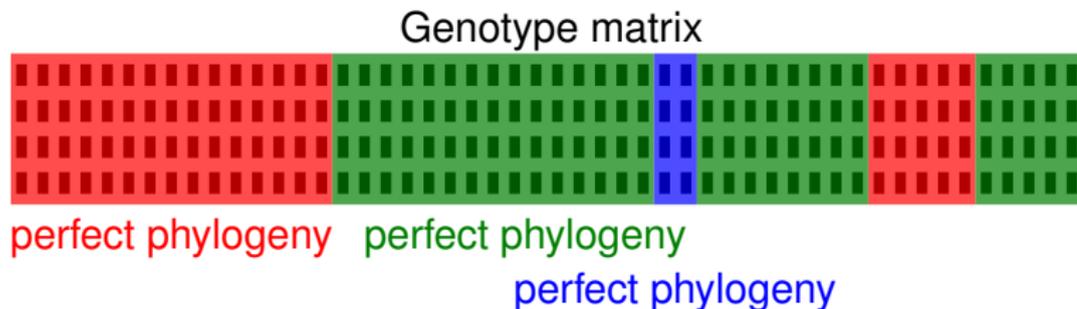
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Objective of the integrated approach.

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The formal computational problem.

We are interested in the computational complexity of
the function χ_{PP} :

- It gets genotype matrices as input.
- It maps them to a number k .
- This number is minimal such that the sites can be covered by k sets, each admitting a perfect phylogeny.
(We call this a **pp-partition**.)

Finding pp-partitions of haplotype matrices.

We start with a special case:

- The inputs M are **already haplotype matrices**.
- The inputs M **do not allow a perfect phylogeny**.
- What is $\chi_{PP}(M)$?

Example

M :

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 |

No perfect phylogeny is possible.

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Perfect phylogeny

Perfect phylogeny

$$\chi_{PP}(M) = 2.$$

Bad news about pp-partitions of haplotype matrices.

Theorem

Finding *optimal pp-partition of haplotype matrices* is equivalent to finding *optimal graph colorings*.

Proof sketch for first direction.

- 1 Let G be a graph.
- 2 Build a matrix with a column for each vertex of G .
- 3 For each edge of G add four rows inducing the submatrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$.
- 4 The submatrix enforces that the columns lie in different perfect phylogenies. □

Implications for pp-partitions of haplotype matrices.

Corollary

If $\chi_{PP}(M) = 2$ for a haplotype matrix M , we can find an optimal pp-partition in polynomial time.

Corollary

Computing χ_{PP} for haplotype matrices is

- *NP-hard,*
- *not fixed-parameter tractable, unless $P = NP$,*
- *very hard to approximate.*

Finding pp-partitions of genotype matrices.

Now comes the general case:

- The inputs M are **genotype matrices**.
- The inputs M **do not allow a perfect phylogeny**.
- What is $\chi_{PP}(M)$?

Example

M :

| | | | |
|---|---|---|---|
| 2 | 2 | 2 | 2 |
| 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 2 | 2 | 0 |
| 1 | 1 | 0 | 0 |

No perfect phylogeny is possible.

Finding pp-partitions of genotype matrices.

Now comes the general case:

- The inputs M are **genotype matrices**.
- The inputs M **do not allow a perfect phylogeny**.
- What is $\chi_{PP}(M)$?

Example

M :

| | | | |
|---|---|---|---|
| 2 | 2 | 2 | 2 |
| 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 2 | 2 | 0 |
| 1 | 1 | 0 | 0 |

Perfect phylogeny

Perfect phylogeny

$$\chi_{PP}(M) = 2.$$

Bad news about pp-partitions of haplotype matrices.

Theorem

Finding *optimal pp-partition of genotype matrices* is at least as hard as finding *optimal colorings of 3-uniform hypergraphs*.

Proof sketch.

- 1 Let G be a 3-uniform hypergraph.
- 2 Build a matrix with a column for each vertex of G .
- 3 For each hyperedge of G add four rows inducing the submatrix $\begin{pmatrix} 2 & 2 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- 4 The submatrix enforces that the three columns do not all lie in the same perfect phylogeny. □

Implications for pp-partitions of genotype matrices.

Corollary

Even if we know $\chi_{PP}(M) = 2$ for a genotype matrix M , finding a pp-partition of any fixed size is still

- NP-hard,
- not fixed-parameter tractable, unless $P = NP$,
- very hard to approximate.

Automatic optimal pp-partitioning is hopeless, but...

- The hardness results are **worst-case** results for **highly artificial inputs**.
- **Real biological data** might have special properties that make the problem **tractable**.
- One such property is that perfect phylogenies are often perfect **path** phylogenies:
In HapMap data, in 70% of the blocks where a perfect phylogeny is possible a perfect path phylogeny is also possible.

Example of a perfect path phylogeny.

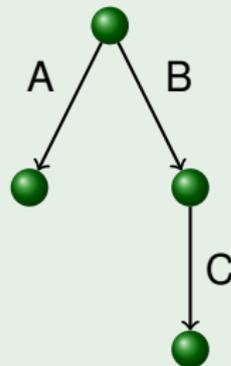
Genotype matrix

| | A | B | C |
|------|---|---|---|
| $G:$ | 2 | 2 | 2 |
| | 0 | 2 | 0 |
| | 2 | 0 | 0 |
| | 0 | 2 | 2 |

Haplotype matrix

| | A | B | C |
|------|---|---|---|
| $H:$ | 1 | 0 | 0 |
| | 0 | 1 | 1 |
| | 0 | 0 | 0 |
| | 0 | 1 | 0 |
| | 0 | 0 | 0 |
| | 1 | 0 | 0 |
| | 0 | 0 | 0 |
| | 0 | 1 | 1 |

Perfect path phylogeny



The modified formal computational problem.

We are interested in the computational complexity of the function χ_{PPP} :

- It gets genotype matrices as input.
- It maps them to a number k .
- This number is minimal such that the sites can be covered by k sets, each admitting a perfect **path** phylogeny. (We call this a ppp-partition.)

Good news about ppp-partitions of genotype matrices.

Theorem

Optimal ppp-partitions of genotype matrices can be computed in polynomial time.

Algorithm

- 1 Build the following partial order:
 - Can one column be above the other in a phylogeny?
 - Can the columns be the two children of the root of a perfect path phylogeny?
- 2 Cover the partial order with as few compatible chain pairs as possible.
For this, a maximal matching in a special graph needs to be computed.

Summary

- Finding optimal pp-partitions is **intractable**.
- It is even intractable to find a pp-partition when **just two noncontiguous blocks are known to suffice**.
- For perfect **path** phylogenies, optimal partitions can be computed **in polynomial time**.

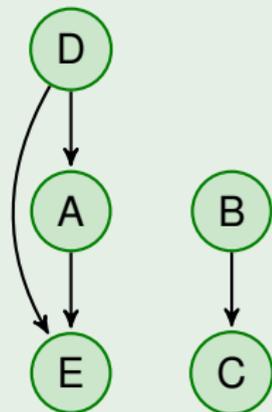
The algorithm in action.

Computation of the partial order.

Genotype matrix

| | A | B | C | D | E |
|------|---|---|---|---|---|
| $G:$ | 2 | 2 | 2 | 2 | 2 |
| | 0 | 1 | 2 | 1 | 0 |
| | 1 | 0 | 0 | 1 | 2 |
| | 0 | 2 | 2 | 0 | 0 |

Partial order



Partial order: \rightarrow

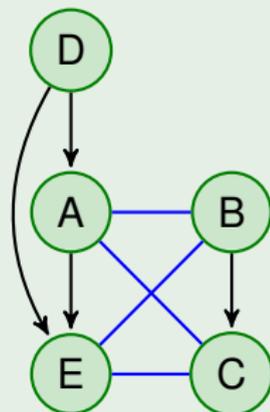
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Computation of the partial order.

Genotype matrix

| | A | B | C | D | E |
|----|---|---|---|---|---|
| | 2 | 2 | 2 | 2 | 2 |
| G: | 0 | 1 | 2 | 1 | 0 |
| | 1 | 0 | 0 | 1 | 2 |
| | 0 | 2 | 2 | 0 | 0 |

Partial order



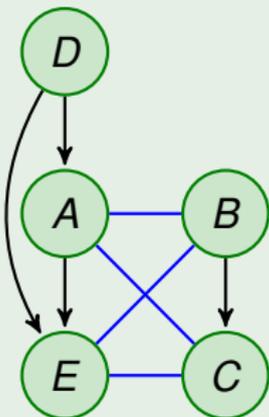
Partial order: →

Compatible as children of root: —

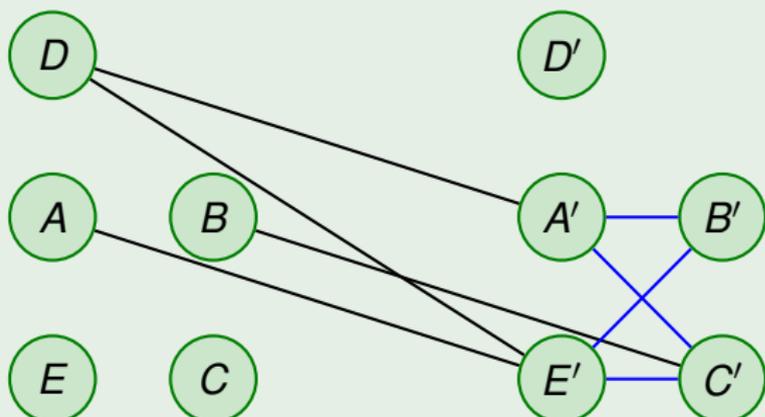
The algorithm in action.

The matching in the special graph.

Partial order

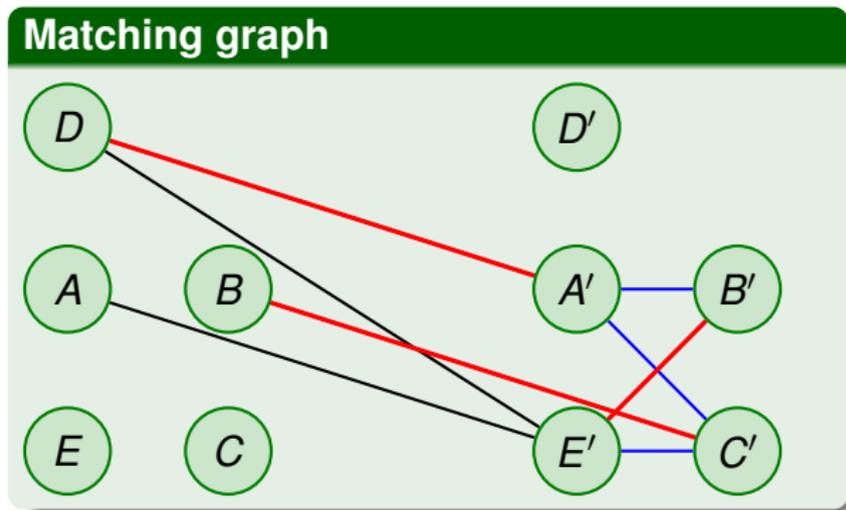
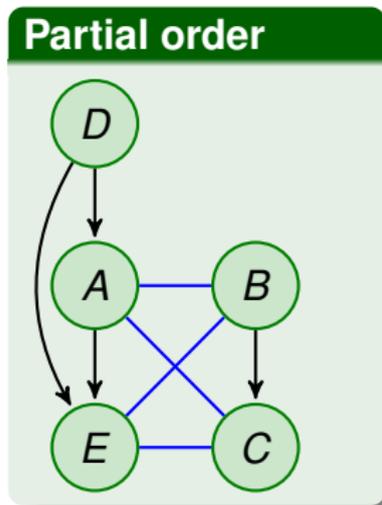


Matching graph



The algorithm in action.

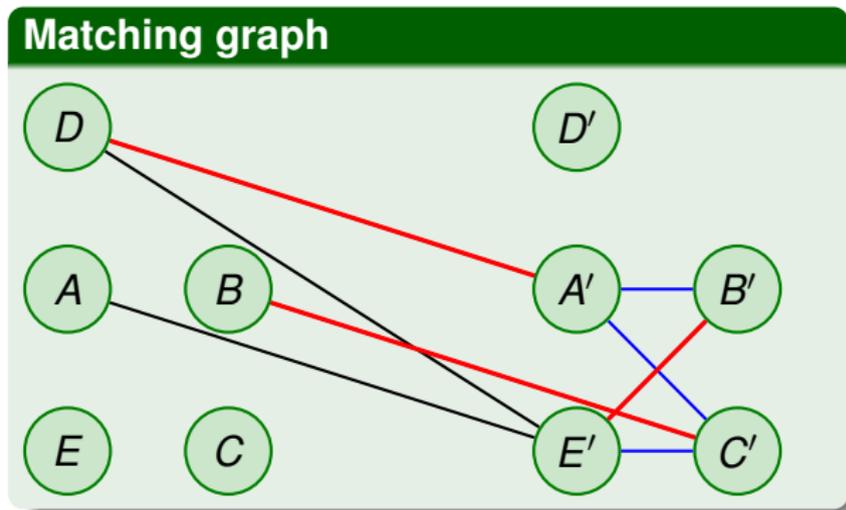
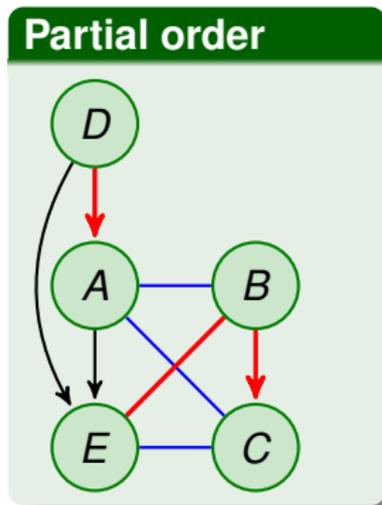
The matching in the special graph.



A **maximal matching** in the matching graph

The algorithm in action.

The matching in the special graph.



A **maximal matching** in the matching graph induces **perfect path phylogenies**.